

Multivariate integration over \mathbb{R}^s with exponential rate of convergence

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Abstract

In this paper we analyze the approximation of multivariate integrals over the Euclidean plane for functions which are analytic. We show explicit upper bounds which attain the exponential rate of convergence. We use an infinite grid with different mesh sizes and lengths in each direction to sample the function, and then truncate it. In our analysis, the mesh sizes and the truncated domain are chosen by optimally balancing the truncation error and the discretization error.

This paper derives results in comparable function space settings, extended to \mathbb{R}^s , as which were recently obtained in the unit cube by Dick, Larcher, Pillichshammer and Woźniakowski (2011), see [1]. They showed that both lattice rules and regular grids, with different mesh sizes in each direction, attain exponential rates, hence motivating us to analyze only cubature formula based on regular meshes. We further also amend the analysis of older publications, e.g., Sloan and Osborn (1987) [8] and Sugihara (1987) [10], using lattice rules on \mathbb{R}^s by taking the truncation error into account and extending them to take the anisotropy of the function space into account.

1 Introduction

We study approximation of the multivariate integral

$$I(f) := \int_{\mathbb{R}^s} f(\mathbf{x}) \, d\mathbf{x} \quad (1)$$

for functions defined over the s -dimensional Euclidean space \mathbb{R}^s , which belong to some normed weighted function space, by using a cubature rule using function values of f , cf. (3). The function space setting will be formally introduced in Section 2, but let us make some remarks already here. We stress that we do not integrate explicitly against any specific probability density function, and as such this implicitly requires the function to decay fast enough towards infinity to be integrable, i.e., at least with a polynomial rate strictly larger than 1, which will be denoted by assumption **(F1)** in Section 2. In particular we are interested in approximating the above integral with an exponential rate $\mathcal{O}(\exp(-C_1(s)N^{C_2(s)}))$ or $\mathcal{O}(\exp(-C_3(s)N^{C_4(s)}(\ln N)^{C_5(s)}))$ in the number of integration nodes N , where $C_1(s)$, $C_2(s)$, $C_3(s)$, $C_4(s)$ and $C_5(s)$ are constants depending on s .

The Fourier transform of a function $f(\mathbf{x})$ will be denoted by

$$\hat{f}(\boldsymbol{\xi}) := \int_{\mathbb{R}^s} f(\mathbf{x}) e^{-2\pi i \boldsymbol{\xi} \cdot \mathbf{x}} \, d\mathbf{x}, \quad \boldsymbol{\xi} \in \mathbb{R}^s.$$

It is well known that the decay of the Fourier transform can be interpreted as a kind of smoothness of the function. In the one dimensional case the Fourier transform of an analytic function decays faster than any polynomial. Moreover, according to the Paley–Weiner theorem if a real-valued analytic function has some continuation into the complex plane and satisfies some decay conditions then its Fourier transform decays exponentially fast, see [9, Theorem 2.1, 3.1]. As a natural extension to the multivariate case, we will therefore further restrict to the case where the Fourier transform decays at least exponential, which will be denoted by assumption **(W2)** in Section 2.

Our analysis is done for a cubature algorithm which can be described by two consecutive steps. First we approximate $I(f)$ by

$$I_{\mathbf{h}}(f) := h_1 \cdots h_s \sum_{\mathbf{k} \in \mathbb{Z}^s} f(k_1 h_1, \dots, k_s h_s), \quad (2)$$

with different mesh sizes h_j , $j = 1, \dots, s$, for different dimensions, based on the anisotropy of the Fourier transform of f , and then we truncate the infinite sum to obtain our final quadrature approximation

$$Q_{\mathbf{h}}^{\mathcal{D}_{\mathbf{n}}}(f) := h_1 \cdots h_s \sum_{\mathbf{k} \in \mathcal{D}_{\mathbf{n}}} f(k_1 h_1, \dots, k_s h_s), \quad (3)$$

for some appropriate truncation set $\mathcal{D}_{\mathbf{n}}$ depending on the anisotropic decay of the function itself. Although this is a straightforward algorithm, it will be essential to determine the mesh sizes h_j , $j = 1, \dots, s$, and the truncation set $\mathcal{D}_{\mathbf{n}}$ in an optimal way to balance their error contributions. We remark that for integration over \mathbb{R}^s there is only little difference between the “rectangle”, “trapezoidal” and “midpoint” rule. Furthermore, as the distinction does not play any role in the further analysis we just denote our rule as the “trapezoidal rule”, similar to what was argued in [14] in the univariate setting. We choose the truncation set $\mathcal{D}_{\mathbf{n}}$ such that $-n_j/2 \leq k_j \leq n_j/2$ for all $j = 1, \dots, s$, and for well chosen truncation points $n_j/2$. This approach leads to two errors which need to be controlled, more specifically, a discretization error and a truncation error:

$$|I(f) - Q_{\mathbf{h}}^{\mathcal{D}_{\mathbf{n}}}(f)| \leq |I(f) - I_{\mathbf{h}}(f)| + |I_{\mathbf{h}}(f) - Q_{\mathbf{h}}^{\mathcal{D}_{\mathbf{n}}}(f)|. \quad (4)$$

The discretization error will be studied in Section 3 and the truncation error will be studied in Section 4. In order to achieve an exponential rate of convergence we will further assume that the final integrand function decays at least exponential, cf. assumption **(F2)** in Section 2, to control the truncation error and we analyze this in Section 4.1. This may seem like an unreasonably strong assumption, but for functions that do not decay this fast we employ a variable transform. E.g., the so-called exponential and double exponential transforms by Takahasi and Mori [12]. This leads us to also study functions which decay double exponentially, cf. assumption **(F3)** in the next section and we analyse its truncation error in Section 4.2. The idea from [12] is to use a suitable change of variables to transform an integral over a domain $\Omega \subseteq \mathbb{R}^s$ into another integral over the Euclidean space \mathbb{R}^s , for which the integrand decays double exponentially fast toward infinity. The transformed integral can then be approximated by the trapezoidal rule.

The extreme accuracy of the trapezoidal rule for the integration of one dimensional analytic periodic functions on intervals and on the real line was studied by many researchers, see [14] for a recent overview by Trefethen and Weideman. This is easy to show on periodic intervals. As a hand-waving argument the analysis can be carried over to integration on \mathbb{R} if the function goes to zero fast and smooth enough, as then the truncated integrand is “nearly periodic” on the truncated interval. Also in the one-dimensional case this can be quantified in exact terms by an analysis of the discretization and truncation errors.

We now give a quick overview of existing results. In the setting of multivariate integration over the unit cube $[0, 1]^s$, in a weighted periodic function space with exponentially decaying Fourier series expansions, it was recently shown that a regular grid (with different mesh sizes for the different dimensions), as well as “good” rank-1 lattice rules, achieve exponential rates of convergence, see Dick et al [1], and Kritzer et al [4] for the infinite-dimensional case. We note explicitly that, with respect to the rate, it is shown in [1] that there is no difference if one takes a grid or a lattice rule in the case of periodic functions over $[0, 1]^s$. The weighted function space over \mathbb{R}^s in the current paper, see Section 2, is modeled after the weighted function space over $[0, 1]^s$ in these papers.

Multivariate integration over \mathbb{R}^s with exponential convergence rates was also studied in [8, 10] and recently in [3]. In [8, 10], the integral $I(f)$ is approximated by an equal-weight rule based on a infinite lattice. In [8], Sloan and Osborn also considered a regular grid as a special case of a lattice, a so-called “cubic lattice”, next to more general lattices and in particular the body centered cubic lattice. They studied the case where the function is isotropic, and thus only have one mesh size for all dimensions, and do not discuss how to balance discretization and truncation error. Instead they fix their truncation domain in advance (to be all lattice points inside a sphere around the origin with a fixed radius) and then study the convergence in terms of decreasing the mesh size. Also in their result it makes no difference, in terms of the exponential rate, if one takes a more general lattice as a point set or a regular grid. They illustrate their technique numerically on different examples for fixed truncation radius.

In [3], Irrgeher et al use a weighted tensor product of Gauss–Hermite quadrature rule for integrating a class of analytic functions. In their context, the analyticity of a function is characterized by the exponential decay of its Hermite coefficients. They show an explicit error bound which attain the exponential rate of convergence. Moreover, they study so-called exponential convergence with tractability, i.e., study the dependence of the constants of the exponential rate on the dimension. Finally, they show an important result, i.e., necessary conditions on the decay of Hermite coefficients such that the constants become independent of the dimension.

We write $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \{0, 1, 2, \dots\}$. We use the shorthand notation $\{1 : s\} := \{1, \dots, s\}$. Further we write $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$.

2 Weighted function spaces

First, we introduce a function space in the univariate case. Let $\nu(x)$ and $\omega(\xi)$ be strictly positive functions. We then define a one-dimensional space of integrable functions $E(\nu, \omega)$ as follows

$$E(\nu, \omega) := \left\{ f : \mathbb{R} \rightarrow \mathbb{R} : \|f\| := \|f(x) \nu(x)\|_{L^\infty(\mathbb{R})} + \|\hat{f}(\xi) \omega(\xi)\|_{L^\infty(\mathbb{R})} < \infty \right\}.$$

The idea is that $\nu(x)$ controls the decay of the function and $\omega(\xi)$ controls the decay of the Fourier transform of the function. The faster $\nu(x)$ and $\omega(\xi)$ increase for increasing x and ξ , the faster the the function values and its Fourier transform decay.

For the multivariate case, we consider the tensor product. Let $\nu : \mathbb{R}^s \rightarrow \mathbb{R}_+$ and $\omega : \mathbb{R}^s \rightarrow \mathbb{R}_+$. Then we define

$$E_s(\nu, \omega) := \left\{ f : \mathbb{R}^s \rightarrow \mathbb{R} : \|f\| := \|f(\mathbf{x}) \nu(\mathbf{x})\|_{L^\infty(\mathbb{R}^s)} + \|\hat{f}(\boldsymbol{\xi}) \omega(\boldsymbol{\xi})\|_{L^\infty(\mathbb{R}^s)} < \infty \right\}.$$

In this paper we will consider two main kinds of decay of the Fourier transform:

(W1) decay with polynomial order, for any $\alpha > 0$:

$$\omega(\boldsymbol{\xi}) = \prod_{j=1}^s (1 + |\xi_j|^{1+\alpha}),$$

(W2) exponentially decay, with all $a_j, b_j \geq 0$:

$$\omega(\xi) = \exp \left(\sum_{j=1}^s a_j |\xi_j|^{b_j} \right).$$

Likewise we will assume three main kinds of decay on the function values:

(F1) polynomial decay, for any $\alpha > 0$:

$$\nu(\mathbf{x}) = \prod_{j=1}^s (1 + |x_j|^{1+\alpha}),$$

(F2) exponential decay, with all $c_j, d_j \geq 0$:

$$\nu(\mathbf{x}) = \exp \left(\sum_{j=1}^s c_j |x_j|^{d_j} \right),$$

(F3) double exponentially decay, with all $c_j, d_j, e_j \geq 0$:

$$\nu(\mathbf{x}) = \exp \left(\sum_{j=1}^s e_j \exp(c_j |x_j|^{d_j}) \right).$$

The double exponential decay **(F3)** of the function was introduced by Takahasi and Mori [12]. They studied one dimensional integration with singularities at the endpoints. By a suitable change of variable they transformed the integration into integration over the real line where the transformed integrand decays double exponentially fast and they then applied a trapezoidal rule to approximate the integration over the real line.

The above univariate function space $E(\nu, \omega)$ where $\omega(\xi)$ satisfies assumption **(W2)** and assumption **(F2)** or **(F3)** with $d_1 = 1$ is studied in [11]. It was shown that the trapezoidal rule is nearly optimal, in the sense that the upper bound and the lower bound of the error are nearly equal where the difference depends on the number of function evaluations.

We remark that the combination of assumptions **(W1)** and **(F1)** allow the use of the Poisson summation formula [9] which links the discretization of the function to a discretization of its Fourier transform:

$$h_1 \cdots h_s \sum_{\mathbf{k} \in \mathbb{Z}^s} f(k_1 h_1, \dots, k_s h_s) = \sum_{\mathbf{k} \in \mathbb{Z}^s} \hat{f}(k_1/h_1, \dots, k_s/h_s). \quad (5)$$

3 The discretization error

We consider the case when the Fourier transform of the function decays exponentially fast as in **(W2)**. To be able to use the Poisson summation formula we also need the function itself to decay at least polynomially as in **(F1)**. We note that this condition is also needed for integrability. Before we prove the discretization error we need the following lemma.

Lemma 1. For $b > 0$ and positive $h \leq 1$, we have

$$\sum_{k=1}^{\infty} \exp(-k^b/h) \leq \exp(-1/h) \frac{e\Gamma(1/b)}{b}.$$

Proof. We have the estimate

$$\begin{aligned} \sum_{k=1}^{\infty} \exp(-k^b/h) &= \exp(-1/h) \sum_{k=1}^{\infty} \exp(-(k^b - 1)/h) \\ &\leq \exp(-1/h) \sum_{k=1}^{\infty} \exp(-k^b + 1). \end{aligned}$$

where the last inequality yields from the condition $h \leq 1$ and $k^b - 1 \geq 0$. Moreover, interpreting the sum as an infinite right-rectangle rule applied to a decreasing function, we have

$$\sum_{k=1}^{\infty} e^{-k^b} \leq \int_0^{\infty} e^{-x^b} dx = \frac{1}{b} \int_0^{\infty} t^{\frac{1}{b}-1} e^{-t} dt = \frac{\Gamma(1/b)}{b}.$$

Combining these gives the desired result. \square

We are now ready to prove a bound on the discretization error when we choose $h_j = (a_j h)^{1/b_j}$.

Theorem 1. Suppose $f(\mathbf{x})$ belonging to $E_s(\nu, \omega)$ for some $\nu(\boldsymbol{\xi})$ and $\omega(\mathbf{x})$ with conditions **(F1)** and **(W2)** such that for some fixed $a_j, b_j > 0$, we have

$$\omega(\boldsymbol{\xi}) = \exp \left(\sum_{j=1}^s a_j |\xi_j|^{b_j} \right).$$

Then for any $h > 0$ such that $h \leq 1/\ln(2e)$, the following inequality is satisfied

$$|I_{\mathbf{h}}(f) - I(f)| \leq C(s, \omega) \|\hat{f}(\boldsymbol{\xi}) \omega(\boldsymbol{\xi})\|_{L^\infty(\mathbb{R}^s)} \exp(-1/h),$$

where

$$C(s, \omega) = C(s, b_1, \dots, b_s) := 2e \sum_{\emptyset \neq \mathbf{u} \subseteq \{1:s\}} \prod_{j \in \mathbf{u}} \frac{\Gamma(1/b_j)}{b_j},$$

and we set $h_j = (a_j h)^{1/b_j}$.

Proof. We first note the obvious identity $I(f) = \hat{f}(0)$. By the assumptions on the decay of the function values and the decay of the Fourier transform we can use the Poisson summation formula (5) to rewrite (2) and obtain

$$I_{\mathbf{h}}(f) = \sum_{\mathbf{k} \in \mathbb{Z}^s} \hat{f}(k_1/h_1, \dots, k_s/h_s).$$

Thus we have

$$\begin{aligned} |I_{\mathbf{h}}(f) - I(f)| &= \left| \sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}} \hat{f}(k_1/h_1, \dots, k_s/h_s) \frac{\omega(k_1/h_1, \dots, k_s/h_s)}{\omega(k_1/h_1, \dots, k_s/h_s)} \right| \\ &\leq \|\hat{f}(\boldsymbol{\xi}) \omega(\boldsymbol{\xi})\|_{L^\infty(\mathbb{R}^s)} \sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}} \exp \left(- \sum_{j=1}^s a_j |k_j|^{b_j} h_j^{-b_j} \right). \end{aligned}$$

To evenly distribute the leading terms in the sum we fix $h = h_j^{b_j}/a_j$ for each j and then we can bound the sum using Lemma 1 as

$$\begin{aligned}
\sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}} \exp\left(-\sum_{j=1}^s |k_j|^{b_j}/h\right) &= -1 + \prod_{j=1}^s \left(1 + 2 \sum_{k=1}^{\infty} \exp(-k^{b_j}/h)\right) \\
&\leq -1 + \prod_{j=1}^s \left(1 + \exp(-1/h) \frac{2e}{b_j} \Gamma\left(\frac{1}{b_j}\right)\right) \\
&= \sum_{\emptyset \neq \mathbf{u} \subseteq \{1:s\}} (2e \exp(-1/h))^{|u|} \prod_{j \in \mathbf{u}} \frac{\Gamma(1/b_j)}{b_j} \\
&= 2e \exp(-1/h) \sum_{\emptyset \neq \mathbf{u} \subseteq \{1:s\}} (2e \exp(-1/h))^{|u|-1} \prod_{j \in \mathbf{u}} \frac{\Gamma(1/b_j)}{b_j} \\
&\leq 2e \exp(-1/h) \sum_{\emptyset \neq \mathbf{u} \subseteq \{1:s\}} \prod_{j \in \mathbf{u}} \frac{\Gamma(1/b_j)}{b_j},
\end{aligned}$$

where for the last inequality we used $h \leq 1/\ln(2e)$. \square

4 The truncation error

We consider the following truncation domain,

$$\mathcal{D}_{\mathbf{n}} = \{\mathbf{k} \in \mathbb{Z}^s : -n_j/2 \leq k_j \leq n_j/2 \text{ for all } j = 1, \dots, s\},$$

where $n_j \in \mathbb{N}_0$, $j = 1, \dots, s$, and $n_j/2$ are called the truncation points. Without loss of generality we consider the case where n_1, \dots, n_s are even integers. The total number of function evaluations is then given by

$$n = (n_1 + 1) \cdots (n_s + 1).$$

We could choose the truncation points differently for the left hand side and the right hand side. However we assume it is easy to use a centering mapping, e.g., as in [7], to transform the function into a new function with symmetric decay around the origin. From the bound (4) it is clear that if the function decays not at least exponentially at infinity then our proposed algorithm is not efficient. For example, for the one dimensional function $f(x) = (1 + x^2)^{-1}$, it will take thousands of function evaluations to achieve a few digits of accuracy, see [14]. In order to achieve an exponential convergence rate it is required that the function decays at least exponentially fast as in assumption **(F2)**. We consider this case in Section 4.1.

For functions that are smooth enough in the sense of assumption **(W2)** but unfortunately decay slower, i.e., decay at most of polynomial order **(F1)**, we propose to use a variable transformation such that the function decays at least exponential **(F2)**, or even double exponential **(F3)**. This will be studied in Section 4.2.

4.1 Functions that decay exponentially fast

The following lemma will be useful in the further part.

Lemma 2. For $c > 0$, $d \geq 1$ and $n \geq 0$, we have

$$\sum_{k=n}^{\infty} \exp(-c k^d) \leq \exp(-c n^d) \left(1 + \frac{\Gamma(1/d)}{c^{1/d} d} \right).$$

Proof. We have

$$\begin{aligned} \sum_{k=n}^{\infty} \exp(-c k^d) &= \exp(-c n^d) \sum_{k=n}^{\infty} \exp(-c (k^d - n^d)) \\ &\leq \exp(-c n^d) \sum_{k=n}^{\infty} \exp(-c (k - n)^d) \\ &= \exp(-c n^d) \sum_{k=0}^{\infty} \exp(-c k^d) \\ &= \exp(-c n^d) \left(1 + \sum_{k=1}^{\infty} \exp(-c k^d) \right) \\ &\leq \exp(-c n^d) \left(1 + \int_0^{\infty} \exp(-c x^d) dx \right) \\ &= \exp(-c n^d) \left(1 + \frac{\Gamma(1/d)}{c^{1/d} d} \right). \end{aligned}$$

The first inequality is due to the following inequality

$$x^d - n^d \geq (x - n)^d, \quad (6)$$

for any $d \geq 1$ and $x \geq n \geq 0$.

Indeed, we consider the following function

$$f(x) = x^d - n^d - (x - n)^d.$$

Since

$$\frac{df}{dx} = d x^{d-1} - d (x - n)^{d-1} \geq 0,$$

for every $x \geq n \geq 0$. Moreover $f(n) = 0$ then the inequality (6) is easily yielded from the increase of $f(x)$. \square

The truncation error for the function that decays exponentially fast will be estimated by the following theorem.

Theorem 2. Suppose $f(\mathbf{x})$ belonging to $E_s(\nu, \omega)$ for $\nu(\boldsymbol{\xi})$ with condition **(F2)** such that for some fixed $c_j > 0$ and $d_j \geq 1$, we have

$$\nu(\mathbf{x}) = \exp \left(\sum_{j=1}^s c_j |x_j|^{d_j} \right),$$

Then for any $h_j > 0$ and $n_j \geq 0$, for $j = 1, \dots, s$, the following inequality is satisfied

$$|I_{\mathbf{h}}(f) - Q_{\mathbf{h}}^{\mathcal{Q}^n}(f)| \leq C(s, \nu, \mathbf{h}) \|f(\mathbf{x}) \nu(\mathbf{x})\|_{L^\infty(\mathbb{R}^s)} \exp \left(- \inf_j c_j h_j^{d_j} \left(\frac{n_j + 1}{2} \right)^{d_j} \right), \quad (7)$$

where

$$C(s, \nu, \mathbf{h}) := 2^s \prod_{j=1}^s \left(h_j + 2 \frac{\Gamma(1/d_j)}{d_j c_j^{1/d_j}} \right). \quad (8)$$

Proof. From (2) and (3) and the definition of the norm using $\nu(\mathbf{x})$, we have

$$\begin{aligned} |I_{\mathbf{h}}(f) - Q_{\mathbf{h}}^{\mathcal{D}_n}(f)| &= \left| h_1 \cdots h_s \sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \mathcal{D}_n} f(k_1 h_1, \dots, k_s h_s) \right| \\ &\leq h_1 \cdots h_s \sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \mathcal{D}_n} \left| f(k_1 h_1, \dots, k_s h_s) \frac{\nu(k_1 h_1, \dots, k_s h_s)}{\nu(k_1 h_1, \dots, k_s h_s)} \right| \\ &\leq \|f(\mathbf{x}) \nu(\mathbf{x})\|_{L^\infty(\mathbb{R}^s)} h_1 \cdots h_s \sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \mathcal{D}_n} \exp \left(- \sum_{j=1}^s c_j |k_j h_j|^{d_j} \right). \end{aligned} \quad (9)$$

Now we evaluate the sum in the last inequality. We have

$$\sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \mathcal{D}_n} \exp \left(- \sum_{j=1}^s c_j |k_j h_j|^{d_j} \right) \leq 2 \sum_{i=1}^s \left(\sum_{k_1 \in \mathbb{Z}} \cdots \sum_{k_i = \frac{n_i}{2} + 1}^{\infty} \cdots \sum_{k_s \in \mathbb{Z}} \exp \left(- \sum_{j=1}^s c_j |k_j h_j|^{d_j} \right) \right). \quad (10)$$

For each of the $i = 1, \dots, s$, since $d_i \geq 1$, then using Lemma 2, we have

$$\begin{aligned} \sum_{k_i = \frac{n_i}{2} + 1}^{\infty} \exp \left(-c_i k_i^{d_i} h_i^{d_i} \right) &\leq \exp \left(-c_i h_i^{d_i} \left(\frac{n_i}{2} + 1 \right)^{d_i} \right) \left(1 + \frac{\Gamma(1/d_i)}{d_i c_i^{1/d_i} h_i} \right) \\ &\leq \exp \left(-c_i h_i^{d_i} \left(\frac{n_i + 1}{2} \right)^{d_i} \right) \left(1 + 2 \frac{\Gamma(1/d_i)}{d_i c_i^{1/d_i} h_i} \right). \end{aligned} \quad (11)$$

Moreover, we have

$$\begin{aligned} \sum_{k_i \in \mathbb{Z}} \exp \left(-c_i h_i^{d_i} |k_i|^{d_i} \right) &= 1 + 2 \sum_{k_i=1}^{\infty} \exp \left(-c_i h_i^{d_i} k_i^{d_i} \right) \\ &\leq 1 + 2 \int_0^{\infty} \exp \left(-c_i h_i^{d_i} x^{d_i} \right) dx \\ &= 1 + 2 \frac{\Gamma(1/d_i)}{d_i c_i^{1/d_i} h_i}. \end{aligned} \quad (12)$$

Applying (11) and (12) into (10), we get

$$\begin{aligned} \sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \mathcal{D}_n} \exp \left(- \sum_{j=1}^s c_j |k_j h_j|^{d_j} \right) \\ \leq 2 \sum_{i=1}^s \exp \left(-c_i h_i^{d_i} \left(\frac{n_i + 1}{2} \right)^{d_i} \right) \prod_{j=1}^s \left(1 + 2 \frac{\Gamma(1/d_j)}{d_j c_j^{1/d_j} h_j} \right) \end{aligned}$$

$$\leq 2s \exp \left(- \inf_i c_i h_i^{d_i} \left(\frac{n_i + 1}{2} \right)^{d_i} \right) \prod_{j=1}^s \left(1 + 2 \frac{\Gamma(1/d_j)}{d_j c_j^{1/d_j} h_j} \right). \quad (13)$$

Applying (13) into (9), the claim follows. \square

Now we prove the first main result. Let's denote

$$\begin{aligned} B(s) &:= \sum_{j=1}^s \frac{1}{b_j}, \\ D(s) &:= \sum_{j=1}^s \frac{1}{d_j}, \\ C_* &:= \inf_j C_j := \inf_j \frac{c_j^{1/d_j} a_j^{1/b_j}}{2}, \end{aligned} \quad (14)$$

and

$$C := \inf_j \left(\frac{C_*}{2} \right)^{d_j}. \quad (15)$$

Theorem 3. Suppose $f(\mathbf{x})$ belonging to $E_s(\nu, \omega)$ for some $\nu(\boldsymbol{\xi})$ and $\omega(\mathbf{x})$ with conditions **(W2)** and **(F2)** such that for some fixed $a_j, b_j, c_j > 0$ and $d_j \geq 1$, we have

$$\nu(\mathbf{x}) = \exp \left(\sum_{j=1}^s c_j |x_j|^{d_j} \right),$$

and

$$\omega(\boldsymbol{\xi}) = \exp \left(\sum_{j=1}^s a_j |\xi_j|^{b_j} \right).$$

Then for any N sufficiently large, the following inequality is satisfied

$$|I(f) - Q_{\mathbf{h}}^{\mathcal{D}_n}(f)| \leq C(s) \exp \left(-N^{\frac{1}{B(s)+D(s)}} C^{\frac{D(s)}{D(s)+B(s)}} \right) \|f\|,$$

where

$$C(s) := 4s \prod_{j=1}^s \left(a_j^{1/b_j} + 2 \frac{\Gamma(1/d_j)}{d_j c_j^{1/d_j}} \right) + 4e \sum_{\emptyset \neq \mathbf{u} \subseteq \{1:s\}} \prod_{j \in \mathbf{u}} \frac{\Gamma(1/b_j)}{b_j}, \quad (16)$$

we set $h_j = (a_j h)^{1/b_j}$ with

$$h = N^{-\frac{1}{B(s)+D(s)}} C^{\frac{-D(s)}{D(s)+B(s)}}$$

and

$$n_j + 1 = \max \left\{ \left\lfloor \frac{C_*}{C_j} C^{\frac{1}{B(s)+D(s)}} \left(\frac{D(s)}{b_j} - \frac{B(s)}{d_j} \right) N^{\frac{1}{D(s)+B(s)}} \left(\frac{1}{d_j} + \frac{1}{b_j} \right) \right\rfloor, 1 \right\} \quad (17)$$

and N is an upper bound for the total number of function evaluations as

$$(n_1 + 1) \cdots (n_s + 1) \leq N.$$

Proof. We have

$$|I(f) - Q_{\mathbf{h}}^{\mathcal{Q}^n}(f)| \leq |I(f) - I_{\mathbf{h}}(f)| + |I_{\mathbf{h}}(f) - Q_{\mathbf{h}}^{\mathcal{Q}^n}(f)|.$$

We first study the discretization error $|I(f) - I_{\mathbf{h}}(f)|$. We see that all the conditions of Theorem 1 are satisfied. We set $h_j = (a_j h)^{1/b_j}$, $j = 1, \dots, s$, such that the discretization error can be bounded as

$$|I(f) - I_{\mathbf{h}}(f)| \leq C(s, \omega) \exp(-1/h) \|\hat{f}(\boldsymbol{\xi})\omega(\boldsymbol{\xi})\|_{L^\infty(\mathbb{R}^s)}. \quad (18)$$

Now we estimate the truncation error $|I_{\mathbf{h}}(f) - Q_{\mathbf{h}}^{\mathcal{Q}^n}(f)|$. All the conditions of Theorem 2 are satisfied, then substituting the chosen step size into (7), since $h \leq 1$ the truncation error now can be written as

$$|I_{\mathbf{h}}(f) - Q_{\mathbf{h}}^{\mathcal{Q}^n}(f)| \leq C(s, \nu) \exp\left(-\inf_j c_j (a_j h)^{\frac{d_j}{b_j}} \left(\frac{n_j + 1}{2}\right)^{d_j}\right) \|f(\mathbf{x}) \nu(\mathbf{x})\|_{L^\infty(\mathbb{R}^s)}, \quad (19)$$

where

$$C(s, \nu) = 2s \prod_{j=1}^s \left(a_j^{1/b_j} + 2 \frac{\Gamma(1/d_j)}{d_j c_j^{1/d_j}} \right).$$

For all $j = 1, \dots, s$ choose n_j such that

$$n_j + 1 = \max \left\{ \left\lfloor \frac{C_* h^{\frac{B(s)}{D(s)d_j}} N^{\frac{1}{D(s)d_j}}}{C_j h^{1/b_j}} \right\rfloor, 1 \right\}. \quad (20)$$

Note¹ that $\lfloor x \rfloor \geq \frac{x}{2}$, for all $x \geq 1$ then

$$n_j + 1 \geq \frac{C_* h^{\frac{B(s)}{D(s)d_j}} N^{\frac{1}{D(s)d_j}}}{2 C_j h^{1/b_j}}.$$

Applying the above inequality into (19) leads to

$$\begin{aligned} |I_{\mathbf{h}}(f) - Q_{\mathbf{h}}^{\mathcal{Q}^n}(f)| &\leq C(s, \nu) \exp\left(-\inf_j \left(\frac{C_*}{2}\right)^{d_j} \left(h^{B(s)} N\right)^{\frac{1}{D(s)}}\right) \|f(\mathbf{x}) \nu(\mathbf{x})\|_{L^\infty(\mathbb{R}^s)} \\ &= C(s, \nu) \exp\left(-C h^{\frac{B(s)}{D(s)}} N^{\frac{1}{D(s)}}\right) \|f(\mathbf{x}) \nu(\mathbf{x})\|_{L^\infty(\mathbb{R}^s)}. \end{aligned} \quad (21)$$

Adding the discretization error (18) and the truncation error (21) we receive the total error

$$|I(f) - I_{\mathbf{h}}(f)| \leq \tilde{C}(s) \left[\exp(-1/h) + \exp\left(-C h^{\frac{B(s)}{D(s)}} N^{\frac{1}{D(s)}}\right) \right] \|f\|, \quad (22)$$

where $\tilde{C}(s)$ is given by

$$\tilde{C}(s) = 2s \prod_{j=1}^s \left(a_j^{1/b_j} + 2 \frac{\Gamma(1/d_j)}{d_j c_j^{1/d_j}} \right) + 2e \sum_{\emptyset \neq \mathbf{u} \subseteq \{1:s\}} \prod_{j \in \mathbf{u}} \frac{\Gamma(1/b_j)}{b_j}.$$

¹For large x this is an underestimate lower bound for $\lfloor x \rfloor$ since for larger x we have $\lfloor x \rfloor \gg x/2$. There exists a better estimate by considering $\lfloor x \rfloor \geq \lambda x$, for some $0 < \lambda \leq 1$. The following part of the proof is easily changed with respect to this estimate. Furthermore, it is easy to see that choosing a sufficient larger λ we achieve a better constant in the convergence rate. Hence, for large enough x we can choose $\lambda \approx 1$.

It is clear that when h decreases, the first term in the square bracket of (22) increases while the second term decreases. Hence, the optimal h is chosen by balancing these two terms, which leads to

$$h^{-1} = C h^{\frac{B(s)}{D(s)}} N^{\frac{1}{D(s)}},$$

this implies

$$h = N^{\frac{-1}{B(s)+D(s)}} C^{\frac{-D(s)}{D(s)+B(s)}}.$$

This is equivalent to balancing the orders of magnitude of the discretization error (18) and the truncation error (21).

Substituting the chosen h into (20) we get the optimal way of choosing n_1, \dots, n_s as in (17). Thus, from (22) the total error is given by

$$|I(f) - Q_{\mathbf{h}}^{\mathcal{D}_n}(f)| \leq C(s) \exp\left(-N^{\frac{1}{B(s)+D(s)}} C^{\frac{D(s)}{D(s)+B(s)}}\right) \|f\|,$$

for any $N \in \mathbb{N}$, where $C(s) = 2\tilde{C}(s)$.

It is easy to see that $(n_1 + 1) \cdots (n_s + 1) \leq N$ since

$$(n_1 + 1) \cdots (n_s + 1) \leq \prod_{j=1}^s \frac{C_* h^{\frac{B(s)}{D(s)} d_j} N^{\frac{1}{D(s)} d_j}}{C_j h^{1/b_j}} \leq N. \quad \square$$

4.2 Functions that decay double exponentially fast

We will need the result of the following lemma.

Lemma 3. *For $\alpha > 0$, $c > 0$, $d \geq 1$ and $n \geq 0$, we have*

$$\sum_{k=n}^{\infty} \exp(-\alpha \exp(c k^d)) \leq e^{-\alpha} \exp(-\alpha \exp(c n^d)) \left(1 + \frac{\Gamma(1/d)}{(\alpha c)^{1/d}}\right).$$

Proof. We have

$$\begin{aligned} \sum_{k=n}^{\infty} \exp(-\alpha \exp(c k^d)) &= \exp(-\alpha \exp(c n^d)) \sum_{k=n}^{\infty} \exp(-\alpha (\exp(c k^d) - \exp(c n^d))) \\ &\leq \exp(-\alpha \exp(c n^d)) \sum_{k=n}^{\infty} \exp(-\alpha \exp(c (k - n)^d)) \\ &= \exp(-\alpha \exp(c n^d)) \sum_{k=0}^{\infty} \exp(-\alpha \exp(c k^d)), \end{aligned} \quad (23)$$

where the first inequality is due to the following inequality

$$e^{x^d} - e^{n^d} - e^{(x-n)^d} \geq 0,$$

for $d \geq 1$ and $x \geq n \geq 0$. This inequality is easily yielded from the increasing of the following function

$$f(x) = e^{x^d} - e^{n^d} - e^{(x-n)^d}.$$

Indeed, for $d \geq 1$ and $x \geq n \geq 0$

$$\frac{df}{dx} = d x^{d-1} e^{x^d} - d(x-n)^{d-1} e^{(x-n)^d} \geq d x^{d-1} e^{x^d} - d(x-n)^{d-1} e^{x^d} \geq 0.$$

Moreover, we have

$$\begin{aligned} \sum_{k=0}^{\infty} \exp(-\alpha \exp(c k^d)) &\leq e^{-\alpha} + \int_0^{\infty} \exp(-\alpha \exp(c x^d)) \, dx \\ &\leq e^{-\alpha} + \int_0^{\infty} \exp(-\alpha(c x^d + 1)) \, dx \\ &= e^{-\alpha} \left(1 + \frac{\Gamma(1/d)}{(\alpha c)^{1/d}} \right), \end{aligned} \quad (24)$$

where we use $e^x \geq x + 1$, for $x \geq 0$ in the second inequality.

Applying the above estimation into (23), the claim follows. \square

Let's denote

$$e_* = \inf_j e_j.$$

The truncation error is bounded by the following theorem.

Theorem 4. Suppose $f(\mathbf{x})$ belonging to $E_s(\nu, \omega)$ for $\nu(\mathbf{x})$ with condition **(F2)** such that for some fixed $c_j > 0$ and $d_j \geq 1$, we have

$$\nu(\mathbf{x}) = \exp \left(\sum_{j=1}^s e_j \exp(c_j |x_j|^{d_j}) \right),$$

Then for any $h_j > 0$ and $n_j \geq 0$, for $j = 1, \dots, s$, the following inequality is satisfied

$$|I_{\mathbf{h}}(f) - Q_{\mathbf{h}}^{\mathcal{D}_n}(f)| \leq C'(s, \nu, \mathbf{h}) \exp \left[-e_* \exp \left(\inf_j c_j h_j^{d_j} \left(\frac{n_j + 1}{2} \right)^{d_j} \right) \right], \quad (25)$$

where

$$C'(s, \nu, \mathbf{h}) := 2s \prod_{j=1}^s e^{-e_j} \left(h_j + 2 \frac{\Gamma(1/d_j)}{e_j c_j^{1/d_j} d_j} \right).$$

Proof. Using the definition of the norm and the assumption on $\nu(\mathbf{x})$, we get

$$\begin{aligned} |I_{\mathbf{h}}(f) - I_{\mathbf{h}}^n(f)| &= \left| h_1 \cdots h_s \sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \mathcal{D}_n} f(k_1 h_1, \dots, k_s h_s) \right| \\ &\leq h_1 \cdots h_s \sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \mathcal{D}_n} \left| f(k_1 h_1, \dots, k_s h_s) \frac{\nu(k_1 h_1, \dots, k_s h_s)}{\nu(k_1 h_1, \dots, k_s h_s)} \right| \\ &\leq \|f(\mathbf{x}) \nu(\mathbf{x})\|_{L^\infty(\mathbb{R}^s)} h_1 \cdots h_s \sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \mathcal{D}_n} \exp \left[- \sum_{j=1}^s e_j \exp(c_j |k_j h_j|^{d_j}) \right]. \end{aligned} \quad (26)$$

Now we evaluate the summation in the last inequality. We have

$$\begin{aligned} \sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \mathcal{D}_n} \exp \left(- \sum_{j=1}^s e_j \exp (c_j |k_j h_j|^{d_j}) \right) \\ \leq 2 \sum_{i=1}^s \left[\sum_{k_1 \in \mathbb{Z}} \cdots \sum_{k_i = \frac{n_i}{2} + 1}^{\infty} \cdots \sum_{k_s \in \mathbb{Z}} \exp \left(- \sum_{j=1}^s e_j \exp (c_j |k_j h_j|^{d_j}) \right) \right]. \end{aligned} \quad (27)$$

Since $d_i \geq 1$ then using Lemma 3 and $n_i/2 + 1 \geq (n_i + 1)/2$, we have

$$\sum_{k_i = \frac{n_i}{2} + 1}^{\infty} \exp \left(-e_i \exp \left(c_i k_i^{d_i} h_i^{d_i} \right) \right) \leq \exp \left[-e_i \exp \left(c_i h_i^{d_i} \left(\frac{n_i + 1}{2} \right)^{d_i} \right) \right] e^{-e_i} \left(1 + \frac{\Gamma(1/d_i)}{e_i c_i^{1/d_i} h_i d_i} \right). \quad (28)$$

Moreover, using the inequality (24), we obtain

$$\begin{aligned} \sum_{k_i \in \mathbb{Z}} \exp \left(-e_i \exp \left(c_i h_i^{d_i} k_i^{d_i} \right) \right) &= 2 \sum_{k_i=0}^{\infty} \exp \left(-e_i \exp \left(c_i h_i^{d_i} k_i^{d_i} \right) \right) - e^{-e_i} \\ &\leq e^{-e_i} \left(1 + 2 \frac{\Gamma(1/d_i)}{e_i c_i^{1/d_i} h_i d_i} \right). \end{aligned} \quad (29)$$

Applying (28) and (29) into (27), we get

$$\begin{aligned} \sum_{\mathbf{k} \in \mathbb{Z}^s \setminus \mathcal{D}_n} \exp \left[- \sum_{j=1}^s e_j \exp (c_j |k_j h_j|^{d_j}) \right] \\ \leq 2 \sum_{i=1}^s \exp \left[-e_i \exp \left(c_i h_i^{d_i} \left(\frac{n_i + 1}{2} \right)^{d_i} \right) \right] \prod_{j=1}^s e^{-e_j} \left(1 + 2 \frac{\Gamma(1/d_j)}{e_j c_j^{1/d_j} h_j d_j} \right) \\ \leq 2 s \exp \left[-e_* \exp \left(\inf_i c_i h_i^{d_i} \left(\frac{n_i + 1}{2} \right)^{d_i} \right) \right] \prod_{j=1}^s e^{-e_j} \left(1 + 2 \frac{\Gamma(1/d_j)}{e_j c_j^{1/d_j} h_j d_j} \right). \end{aligned} \quad (30)$$

Using the above inequality into (26) yields the claim. \square

Let's use the same notations for C_* as in (14) and C as in (15). Now we obtain the total error for functions which decay double exponential.

Theorem 5. Suppose $f(\mathbf{x})$ belonging to $E_s(\nu, \omega)$ for some $\nu(\boldsymbol{\xi})$ and $\omega(\mathbf{x})$ with conditions **(W2)** and **(F3)** such that for some fixed $a_j, b_j, c_j, e_j > 0$ and $d_j \geq 1$

$$\omega(\boldsymbol{\xi}) = \exp \left(\sum_{j=1}^s a_j |\xi_j|^{b_j} \right),$$

and

$$\nu(\mathbf{x}) = \exp \left(\sum_{j=1}^s e_j \exp (c_j |x_j|^{d_j}) \right),$$

Then for any N sufficiently large, the following inequality is satisfied

$$|I(f) - Q_{\mathbf{h}}^{\mathcal{D}^n}(f)| \leq C'(s, N) \exp \left(-N^{1/B(s)} \left(\ln(e_*^{-B(s)} N) \right)^{-D(s)/B(s)} C^{D(s)/B(s)} B(s)^{D(s)/B(s)} \right) \|f\|,$$

where

$$C'(s) := 4s \prod_{j=1}^s e^{-e_j} \left(a_j^{1/b_j} + 2 \frac{\Gamma(1/d_j)}{e_j c_j^{1/d_j} d_j} \right) + 4e \sum_{\emptyset \neq \mathbf{u} \subseteq \{1:s\}} \prod_{j \in \mathbf{u}} \frac{\Gamma(1/b_j)}{b_j},$$

we set $h_j = (a_j h)^{1/b_j}$ with

$$h = N^{-1/B(s)} \left(\frac{\ln(e_*^{-B(s)} N)}{C B(s)} \right)^{D(s)/B(s)}$$

and

$$n_j + 1 = \max \left\{ \left\lfloor \frac{C_* h^{\frac{B(s)}{D(s) d_j}} N^{\frac{1}{D(s) d_j}}}{C_j h^{1/b_j}} \right\rfloor, 1 \right\}$$

and

$$(n_1 + 1) \cdots (n_s + 1) \leq N.$$

Proof. Using Theorem 1 the discretization error does not change and is given by

$$|I(f) - I_{\mathbf{h}}(f)| \leq C(s, \omega) \exp(-1/h) \|\hat{f}(\boldsymbol{\xi}) \omega(\boldsymbol{\xi})\|_{L^\infty(\mathbb{R}^s)}. \quad (31)$$

The truncation error $|I_{\mathbf{h}}(f) - Q_{\mathbf{h}}^{\mathcal{D}^n}(f)|$ is estimated using Theorem 4. We set $h_j = (a_j h)^{1/b_j}$, $j = 1, \dots, s$, such that the discretization error can be bounded as

$$|I_{\mathbf{h}}(f) - Q_{\mathbf{h}}^{\mathcal{D}^n}(f)| \leq C'(s, \nu) \exp \left[-e_* \exp \left(\inf_j c_j a_j^{\frac{d_j}{b_j}} h^{\frac{d_j}{b_j}} \left(\frac{n_j + 1}{2} \right)^{d_j} \right) \right] \|f(\mathbf{x}) \nu(\mathbf{x})\|_{L^\infty(\mathbb{R}^s)}, \quad (32)$$

where

$$C'(s, \nu) := 2s \prod_{j=1}^s e^{-e_j} \left(a_j^{1/b_j} + 2 \frac{\Gamma(1/d_j)}{e_j c_j^{1/d_j} d_j} \right).$$

For all $j = 1, \dots, s$ choose n_j such that

$$n_j + 1 = \max \left\{ \left\lfloor \frac{C_* h^{\frac{B(s)}{D(s) d_j}} N^{\frac{1}{D(s) d_j}}}{C_j h^{1/b_j}} \right\rfloor, 1 \right\}.$$

Note² that $\lfloor x \rfloor \geq \frac{x}{2}$, for all $x \geq 1$, the truncation error is rewritten as

$$|I_{\mathbf{h}}(f) - Q_{\mathbf{h}}^{\mathcal{D}^n}(f)| \leq C'(s, \nu) \exp \left(-e_* \exp \left(C h^{\frac{B(s)}{D(s)}} N^{\frac{1}{D(s)}} \right) \right) \|f(\mathbf{x}) \nu(\mathbf{x})\|_{L^\infty(\mathbb{R}^s)}. \quad (33)$$

²Using the same argument as in Theorem 3, here we also have a better estimation for large x as $\lfloor x \rfloor \geq \lambda x$, for some $0 < \lambda \leq 1$. In case x is very large we choose $\lambda \approx 1$.

Similarly to Theorem 3, by balancing the orders of magnitude of two errors we obtain

$$h^{-1} = e_* \exp \left(C h^{\frac{B(s)}{D(s)}} N^{\frac{1}{D(s)}} \right) \quad (34)$$

which leads to

$$h = N^{-1/B(s)} \left(\frac{D(s) W \left(C \frac{B(s)}{D(s)} e_*^{-B(s)/D(s)} N^{1/D(s)} \right)}{C B(s)} \right)^{D(s)/B(s)},$$

where $W(\cdot)$ is the Lambert-W function.

Here, we will approximately solve the equation (34) which is equivalent to

$$h = N^{-1/B(s)} \left(\frac{-\ln(e_* h)}{C} \right)^{D(s)/B(s)}.$$

If we choose $h \approx N^{-1/B(s)}$, then this implies

$$h = N^{-1/B(s)} \left(\frac{-\ln(e_* N^{-1/B(s)})}{C} \right)^{D(s)/B(s)}.$$

Substituting the chosen h into (31) and (33), the total error is estimated by the combination of these two inequalities. \square

5 Single and double exponential transformations

In this section we recall some possible variable transformations that give single or double exponential decay of the function values from [12, 6, 5, 13, 14]. At the same time they keep the smoothness of the integrand. That is, we seek ϕ , sufficiently smooth, such that

$$I(f) = \int_{\Omega} f(x) dx = \int_{\mathbb{R}} f(\phi(u)) \phi'(u) du,$$

where $\Omega \subseteq \mathbb{R}$ and

$$|f(\phi(u)) \phi'(u)| \lesssim \exp(-|u|).$$

or

$$|f(\phi(u)) \phi'(u)| \lesssim \exp(-\exp(|u|)).$$

Additionally, if the function $f(x)$ has singularities at the endpoints of Ω , then the discussed transformations also transfer the singularities to $\pm\infty$. Thus, the single and double exponential transformations can be applied to an integral with end-point singularities over a finite interval, an integral over the half infinite interval or an integral over Euclidean space with a moderately decaying integrand.

In the multivariate case, different transformations can be applied to different variables:

$$I(f) = \int_{\Omega} f(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^s} f(\phi_1(u_1), \dots, \phi_s(u_s)) \prod_{j=1}^s \phi'_j(u_j) du,$$

where $\Omega \subseteq \mathbb{R}^s$ and the ϕ_j are suitable transformations.

For the integration with a slow decaying function over a bounded interval, two single exponential mappings are proposed in [13, 14]:

$$\Omega = [-1, 1] \quad \begin{array}{l} \text{tanh rule: } x = \tanh(u), \\ \text{erf rule: } x = \text{erf}(u). \end{array}$$

The functions $\tanh(u)$ and $\text{erf}(u)$ both map the infinite interval \mathbb{R} into the finite interval $(-1, 1)$ such that the effect of endpoint singularities is eliminated. But the erf transform is superior to the tanh transform, since it makes the transformed integrand decay more quickly, i.e., at Gaussian rate while the tanh rule makes the transformed integrand decay just at single exponential rate. Another popular single exponential transform is the sinh rule:

$$\Omega = [-\infty, \infty] \quad \text{sinh rule: } x = \sinh(u),$$

which maps \mathbb{R} to \mathbb{R} and is used to improve the decay of the slowly converging integral, e.g., with the integrand decrease only at polynomial rate.

A family of double exponential transforms that are suitable for a variety of domains Ω and integrands was firstly developed by Mori and Takahashi, see [5, 12, 13]. A transform similar to the tanh rule but which gives a double exponential decay is the tanh-sinh rule:

$$\Omega = [-1, 1] \quad \text{tanh-sinh rule: } x = \tanh\left(\frac{\pi}{2} \sinh(u)\right).$$

The double exponential decay of the transformed integrand is implied from the exponential decay of the derivative of the tanh-sinh function. Indeed, we have

$$\phi'(u) = \left(\tanh\left(\frac{\pi}{2} \sinh(u)\right) \right)' = \frac{\pi}{2} \cosh(x) \text{sech}^2\left(\frac{\pi}{2} \sinh(x)\right) \approx \mathcal{O}\left(\exp\left(-\frac{\pi}{2} \exp(|u|)\right)\right).$$

We also list here two transformations with respect two typical integration domains:

$$\begin{array}{ll} \Omega = [0, \infty) & \text{exp-sinh rule: } x = \exp\left(\frac{\pi}{2} \sinh(u)\right), \\ \Omega = [-\infty, \infty] & \text{sinh-sinh rule: } x = \sinh\left(\frac{\pi}{2} \sinh(u)\right), \end{array}$$

where the exp-sinh rule is used for integrating algebraic functions which decay slowly at $+\infty$ and the sinh-sinh rule is for algebraic functions which decay slowly both at $\pm\infty$. For the special case

$$\int_0^\infty g(x) \exp(-x) dx$$

the integrand already decays at single exponential rate at $+\infty$, hence the following double exponential formula is to add one more exponential decay when x goes to $+\infty$ and double exponential decay when x goes to $-\infty$, namely

$$\Omega = [0, \infty] \quad \text{exp-exp rule: } x = \exp(u - \exp(-u)).$$

For the remainder of this section we will consider the Fourier-type integral, introducing suitable double exponential transforms and briefly showing the intuition behind them. The Fourier-type integration is of the form

$$\int_0^\infty g(x) \sin(\omega x) dx,$$

where $g(x)$ is a slow decaying algebraic function, using the exp-sinh rule is not efficient since the transformed integrand might not be analytic in any strip around the real line, this leads to its Fourier transform not decaying fast enough, for example when $g(x) = 1/(1+x^2)$, or the function itself might not decay fast enough, for example when $g(x) = 1/x$. In this case, two double exponential transformations are proposed:

$$x = M\phi_1(u)/\omega := \frac{u}{\omega(1 - \exp(-6 \sinh(u)))},$$

$$x = M\phi_2(u)/\omega := \frac{Mu}{\omega(1 - \exp(-2u - \alpha(1 - e^{-u}) - \beta(e^u - 1)))}, \quad (35)$$

where $\beta = \frac{1}{4}$, $\alpha = \beta/\sqrt{1 + M \log(1 + M)/4\pi}$ and M is chosen depending on h as

$$M = \pi/h,$$

where h is the step size. It is easy to see that $\phi_1'(u)$ and $\phi_2'(u)$ go to $-\infty$ when u goes to $-\infty$, moreover $\phi_1(u)$ and $\phi_2(u)$ go to u when u goes to $+\infty$. Thus, these two transformations guarantee the transformed integrand decay double exponentially at large negative u . For positive large u , the chosen M makes the integrand decay double exponentially fast to the zeros of $\sin(\omega x)$, as a result, it guarantees the transformed integrand to be very small at the large positive trapezoidal points. Indeed, when n goes to $+\infty$, we have

$$\begin{aligned} \sin(M\phi_1(nh)) &\approx \sin(Mnh) = \sin(\pi n) = 0, \\ \sin(M\phi_2(nh)) &\approx \sin(Mnh) = \sin(\pi n) = 0. \end{aligned}$$

Consequently, we can truncate the infinite trapezoidal rule after a moderate small number of terms.

Similarly, for the Fourier-type integral with the cosine function

$$\int_0^\infty g(x) \cos(\omega x) dx,$$

the following transformations are proposed

$$\begin{aligned} \Omega = [0, \infty) \quad x &= M\phi_1(u - \pi/2M)/\omega, \\ x &= M\phi_2(u - \pi/2M)/\omega. \end{aligned}$$

For further details on this particular transformation in one dimensional case we refer to [5, §5.4] and [6].

6 Numerical results

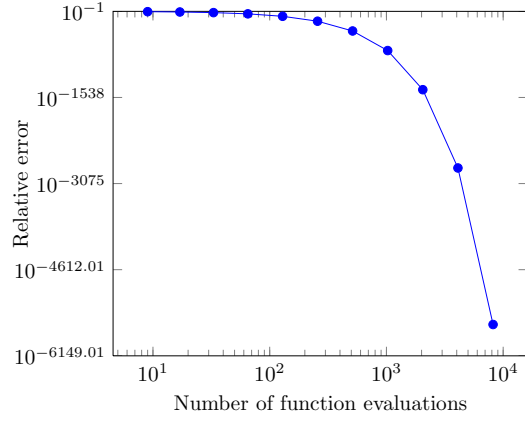
In this section we present some numerical examples to illustrate the results of Theorem 3 and Theorem 5. We consider three toy examples of integration that might be interesting since they are seen in many applications, namely integration with respect to Gaussian distribution, with respect to exponential distribution and Fourier-like integration. In order to calculate with very high precision, our test are implemented in C++ using the *Boost.Multiprecision* library which allows us to define variables with arbitrary decimal digit precision.

Example 1. We consider the integral

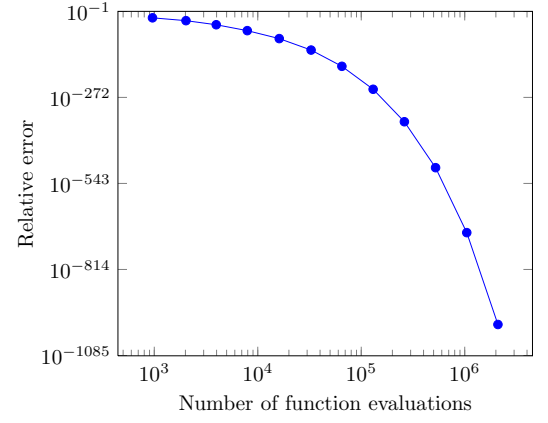
$$I(f) = \int_{\mathbb{R}^s} \exp\left(-\sum_{j=1}^s x_j^2\right) d\mathbf{x} = \prod_{j=1}^s \sqrt{\pi}. \quad (36)$$

The Fourier transform is given as

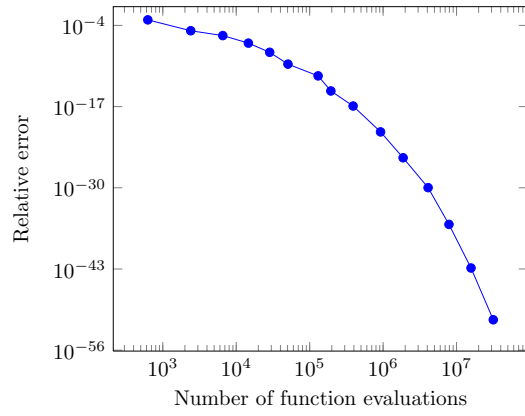
$$\hat{f}(\boldsymbol{\xi}) = \exp\left(-\sum_{j=1}^s \pi^2 \xi_j^2\right) \prod_{j=1}^s 1/2\sqrt{\pi}.$$



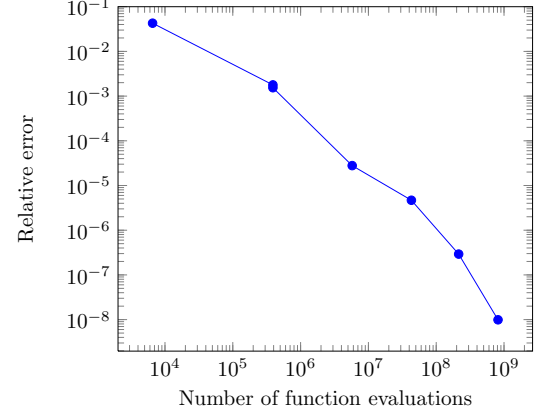
(a) $s = 1$



(b) $s = 2$



(c) $s = 4$



(d) $s = 8$

Figure 1: The relative error of the integration $\int_{\mathbb{R}^s} \exp(-\sum_{j=1}^s x_j^2) d\mathbf{x}$ with respect to the number of function evaluations.

In Figure 1 the relative error $|I(f) - Q_{\mathbf{h}}^{\mathcal{D}^n}(f)|/|I(f)|$ is shown with respect to the number of function evaluations (log-log) where the step sizes are chosen as in Theorem 3 and λ is chosen approximately equal 1. We see that the relative error decays exponentially like $\mathcal{O}(\exp(-N^{1/s}))$.

Example 2. To provide numerical examples that illustrate the convergence rate using double exponential transformation, we consider the integration over \mathbb{R}_+^s with respect to the exponential probability density

$$I(f) = \int_{\mathbb{R}_+^s} f(\mathbf{x}) d\mathbf{x} := \int_{\mathbb{R}_+^s} \prod_{j=1}^s x_j^2 \exp\left(\sum_{j=1}^s -x_j\right) d\mathbf{x}.$$

Since $\exp\left(\sum_{j=1}^s -x_j\right)$ and $g(\mathbf{x}) = \prod_{j=1}^s x_j^2$ are entire functions, due to the Paley–Wiener theorem discussed in Section 1 the Fourier transform $\hat{f}(\boldsymbol{\xi})$ decays exponentially fast.

Using the double exponential change of variable exp-exp rule as

$$x_j = \phi(u_j) := \exp(u_j - \exp(-u_j)),$$

for $j = 1, \dots, s$, results to the integral

$$I(f) = \int_{\mathbb{R}^s} f(\exp(\mathbf{u} - \exp(-\mathbf{u}))) \prod_{j=1}^s (1 + \exp(-u_j)) \exp(u_j - \exp(-u_j)) d\mathbf{u},$$

Applying our proposed method where the step sizes and truncation points are chosen as in Theorem 5, the obtained result is displayed in Figure 2 (log-log). We can see that the convergence rate again matches well with the expected convergence rate in the Theorem 5, i.e., of order $\mathcal{O}(\exp(-c N^{1/s}/\log(N)))$, where c is a constant.

Example 3. In this example we consider a Fourier-type integration

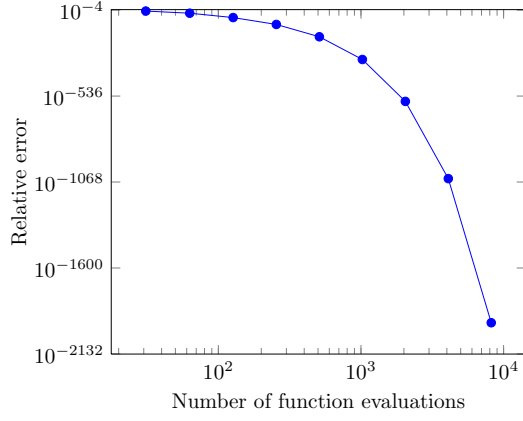
$$I(f) = \int_{\mathbb{R}_+^s} \prod_{j=1}^s \frac{\sin(\omega_j x_j)}{x_j} d\mathbf{x}.$$

We tested this problem numerically up to the 3 dimensional case. We apply double exponential transforms as in (35) and then use our truncated trapezoidal rule. For this oscillating integration instead of truncating the integration domain in advance we use an adaptive truncating strategy, i.e., we sum up only the values of the functions which are greater than a threshold. Figure 3 (log-log) shows the relative error with respect to the number of function evaluations which shows an expected rate of convergence close to $\mathcal{O}(\exp(-c N^{1/s}/\log N))$.

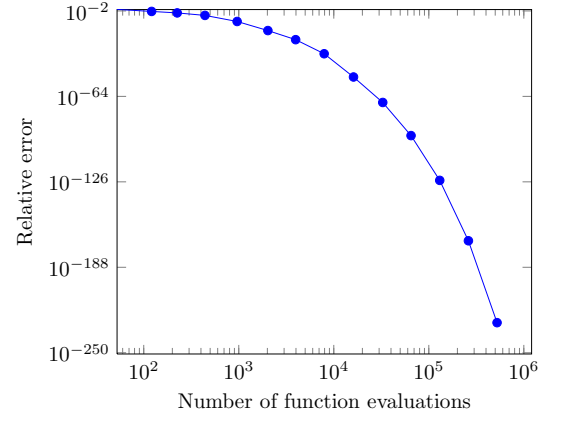
In view of the theoretical results and the results of numerical experiments, we may see the curse of dimensionality. This will be discussed more in Section 7.

7 Conclusion

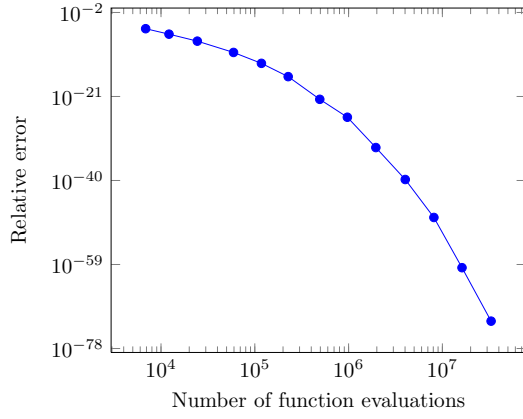
In this work we have analyzed the integration of analytic functions over Euclidean space, we show the truncated trapezoidal rule is efficient, i.e., the error decays exponentially fast. Although somehow restrictive, the proposed method is efficient for functions of a moderate number of



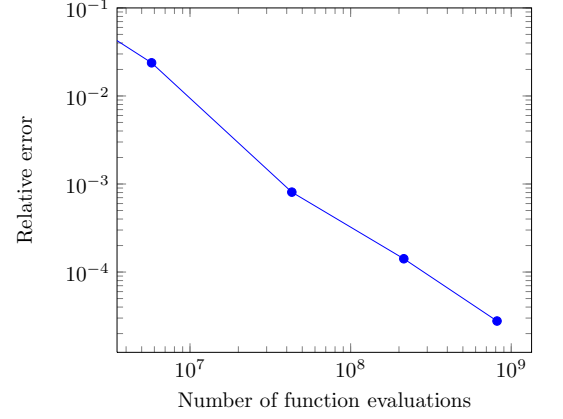
(a) $s = 1$



(b) $s = 2$

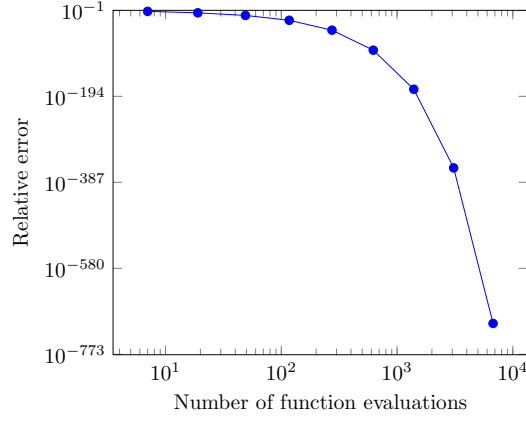


(c) $s = 4$

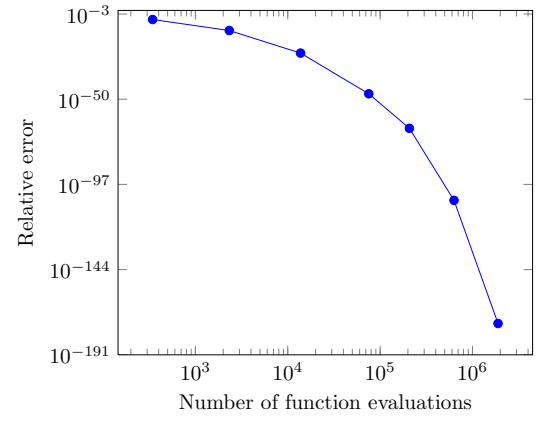


(d) $s = 8$

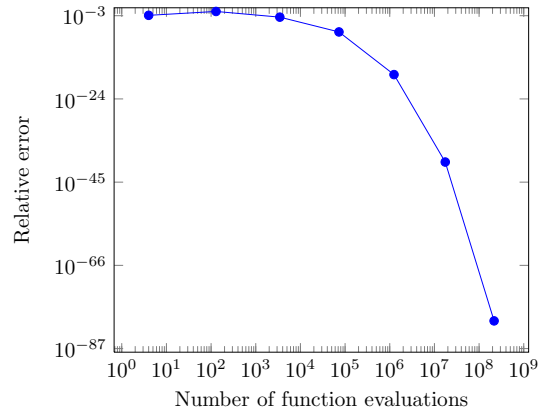
Figure 2: The relative error of the integration $\int_{\mathbb{R}_+^s} \prod_{j=1}^s x_j^2 \exp(-\sum_{j=1}^s x_j) d\mathbf{x}$ with respect to the number of function evaluations.



(a) $s = 1$



(b) $s = 2$



(c) $s = 3$

Figure 3: The relative error of the integration $\int_{\mathbb{R}^s} \prod_{j=1}^s \text{sinc}(x_j) d\mathbf{x}$ with respect to the number of function evaluations.

variables. This setting allows us to estimate both the discretization error and truncation error; and then optimally balance them.

We have also numerically verified our theory by some numerical tests. The obtained results match very well with the theoretical estimations.

Finally, the convergence rates in Theorem 3 and Theorem 5 depend strongly on the dimension s via the parameters $B(s)$ and $D(s)$. One may hope to relax these dependences by making stronger assumptions on the smoothness and the decay of the functions, e.g., by assuming both $B(s)$ and $D(s)$ to be uniformly bounded when s increases. This is equivalent to requiring the function and its Fourier transform to be both concentrated at the origin which is unfortunately in conflict with a general phenomenon called the Heisenberg uncertainty principle [2]. Additionally, although our truncation strategy is the simplest one, easy to be analyzed and implemented, it has not inexhaustibly taken advantage of the trapezoidal rule. For example, for the integration of the Gaussian function as in (36) it is easy to see that it will be more efficient if we truncate the trapezoidal points within a sphere rather than a cube. Therefore, it is necessary to seek for other possible efficient truncation strategies. This will be left for future research.

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